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On anomalies and noncommutative geometry

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(1) Introduction

In the following I discuss examples where basic structures from noncommutative geometry (NCG) [1] naturally arise in quantum field theory. The discussion is based on work with the ultimate aim to get better mathematical understanding of quantum gauge theory models like QCD(3+1). (There is also a close connection with the representation theory of infinite dimensional Lie groups which I shall not discuss.) The examples are restricted to external field problems, i.e. fermions coupled to non–dynamical Yang–Mills fields. This simplification makes possible a complete mathematical analysis. Though rather drastic, it already allows to study in detail several non–trivial aspects of QFT like the structure of UV divergences in the fermion sector and how they lead to anomalies. Moreover, it motivates the development of new efficient calculation tools which, as I believe, should also be useful for analyzing the fully quantized theories.

The philosophy of NCG — to generalize the differential geometric machinery to situations without underlying manifold but rather algebras of Hilbert space operator — seems to be the natural way to understand the relation between the rich differential geometric structure of anomalies (anomalies as de Rham forms, characteristic classes, descendent equations relating anomalies in different dimensions etc.) and their explicit QFT derivation ('dirty' calculations using Feynman diagrams, perturbation theory etc.). A general idea here is to interpret Feynman diagrams as regularized traces of certain operators on some Hilbert space, and to try to identify NCG structures based on the algebra of these operators. The regularized traces are of operators which are not trace class. In the examples discussed anomalies can be identified as regularized traces of commutators $[a,b]=ab-ba$ of certain operators a and b (I believe this is true in general). Even though such an expression is always zero if e.g. a is trace class and b bounded, it can still be defined in more general cases and be non–zero then. Such regularized traces $\mathrm{Tr}_{reg}([a,b])$ are also closely related to the Wodzicki residue and the Dixmier trace playing a fundamental role in NCG [1].

(2) NCG and Schwinger Terms

Graded Differential Algebra (GDA). A basic object in NCG is a GDA generalizing the notion of de Rham forms. To motivate this notion we recall the following purely algebraic characterization of de Rham forms on $\mathbb{R}^d$ (for simplicity we restrict ourselves to manifolds $\mathbb{R}^d$, and all our mappings are C_0^∞, i.e. smooth and compactly supported). One starts with the algebra $\mathcal{C}_d^{(0)} \equiv C_0^\infty(\mathbb{R}^d, \mathrm{gl}_N)$ of $N\times N$–matrix valued functions on $\mathbb{R}^d$. With d the usual exterior differentiation, one defines $\mathcal{C}_d^{(n)}$ as the space of all n–forms which are linear combinations of $\omega_n = X_0 \mathrm{d}X_1\cdots\mathrm{d}X_n$ with $X_i\in\mathcal{C}_d^{(0)}$, and $\mathcal{C}_d = \bigoplus_{n=0}^\infty \mathcal{C}_d^{(n)}$ ($\mathcal{C}_d^{(n)}=\emptyset$ for

$n > d$ here). Then d defines a mapping $\mathcal{C}_d^{(n)} \to \mathcal{C}_d^{(n+1)}$ with $\mathrm{d}^2 = 0$, and

$$\omega_n \omega_m \in \mathcal{C}_d^{(n+m)}, \quad \mathrm{d}(\omega_n \omega_m) = \mathrm{d}(\omega_n)\omega_m + (-)^n \omega_n \mathrm{d}\omega_m \tag{1}$$

$\forall \omega_n \in \mathcal{C}_d^{(n)}$, $\omega_m \in \mathcal{C}_d^{(m)}$, $n, m \in \mathbb{N}_0$. Moreover, there a linear map $\int$ — integration of de Rham forms —

$$\int \omega_n = \begin{cases} \int_{\mathbb{R}^d} \mathrm{tr}_N\,(\omega_n) & \text{for } n = d \\ 0 & \text{otherwise} \end{cases}, \tag{2}$$

(tr_N is the usual trace of $N \times N$–matrices) and Stokes' theorem holds, $\int \mathrm{d}\omega = 0$ for all $\omega \in \mathcal{C}_d$. Such triple $(\mathcal{C}_d, \mathrm{d}, \int)$ is called a GDA.

An important example for a GDA based on algebras of Hilbert space operators is as follows. Consider a separable Hilbert space $\mathcal{H}$ which is decomposed in two orthogonal subspaces, $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. The operator ε which is ± 1 on $\mathcal{H}_\pm$ is a grading operator, $\varepsilon^* = \varepsilon^{-1} = \varepsilon$, and $\mathcal{H}_\pm = \frac{1}{2}(1 \pm \varepsilon)\mathcal{H}$ ($*$ is the Hilbert space adjoint). Denoting as B and B_1 the bounded and trace class operators on $\mathcal{H}$, respectively, and as $B_{2p} = \{a \in B | (a^* a)^p \in B_1\}$ (these are the so–called Schatten classes) one defines the algebras

$$g_p \equiv \{u \in B | [\varepsilon, u] \in B_{2p}\} \tag{3}$$

for $2p$ a positive integer. Then

$$\hat{\omega}_n = (i)^n u_0 [\varepsilon, u_1] \cdots [\varepsilon, u_n] \quad \equiv u_0 \hat{\mathrm{d}} u_1 \cdots \hat{\mathrm{d}} u_n \quad \forall u_i \in g_p, n = 0, 1, \ldots \tag{4}$$

can be regarded as generalized differential forms. Indeed, denoting as $\hat{\mathcal{C}}_p^{(n)}$ the space of all linear combinations of n–forms (4) ($\hat{\mathcal{C}}_p^{(0)} = g_p$),

$$\hat{\mathrm{d}} \hat{\omega}_n = \mathrm{i}(\varepsilon \hat{\omega}_n - (-)^n \hat{\omega}_n \varepsilon) \tag{5}$$

defines a mapping $\hat{\mathcal{C}}_p^{(n)} \to \hat{\mathcal{C}}_p^{(n+1)}$ such that $\hat{\mathrm{d}}^2 = 0$. $\hat{\mathrm{d}}$ can therefore can be regarded as exterior differentiation. One can easily check that the relations (1)–hat hold. An integration $\hat{\int}$ can be defined as

$$\hat{\int} \hat{\omega}_n = \begin{cases} \mathrm{Tr}_C\,(\Gamma^{2p} \hat{\omega}_n) & \text{for } n = 2p - 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

($\Gamma^{2p} = \Gamma\,/\,1$ for $2p$ odd / even) where Γ is a grading operator on $\mathcal{H}$ such that $\varepsilon\Gamma = -\Gamma\varepsilon$ and

$$\mathrm{Tr}_C\,(a) \equiv \frac{1}{2} \mathrm{Tr}\,(a + \varepsilon a \varepsilon) \tag{7}$$

is a conditional Hilbert space trace. Stokes' theorem holds here due to cyclicity of trace.

Then $(\hat{\mathcal{C}}_p, \hat{\mathrm{d}}, \hat{\int})$ with $\hat{\mathcal{C}}_p = \bigoplus_{n=0}^{\infty} \hat{\mathcal{C}}_p^{(n)}$ is a GDA. As discussed below, it actually is a natural generalization of the de Rham complex $(\mathcal{C}_d, \mathrm{d}, \int)$ if $2p = d + 1$.

Quasi–free Second Quantization (QFSQ). The abstract mathematical framework which has been found useful for studying external field problems of fermions is very much in the spirit of NCG, even though historically it has been developed independently. (For a more detailed discussion see e.g. [2]; this approach mainly uses methods from functional analysis [3]. For an alternative approach based on differential geometric methods see [4]. The generalization of the latter from g_1 to $g_{p \geq 1}$ was first given in [5]. My discussion of this is based on [6].)

In an external field problem, the starting point is a 1-particle description of the fermions, and the aim is to 'second quantize' i.e. to find the corresponding QFT description. One has a Hilbert space $\mathcal{H}$ describing the possible 1-particle states, a Hamiltonian H and other observables which are given by self–adjoint operators on $\mathcal{H}$. Note in the following that, even though in most applications $\mathcal{H}$ is a L^2–space over some space manifold, no reference to this manifold or the explicit form of the observables is made, and this makes this framework very general and flexible.

The Hamiltonian H naturally defines a splitting of $\mathcal{H}$ in the subspaces of positive and negative energy states, $\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$. The corresponding grading operator ε with $\varepsilon\mathcal{H}_\pm = \pm\mathcal{H}_\pm$ can be written as $\mathrm{sign}(H)$ (using the spectral theorem of self–adjoint operators where $\mathrm{sign}(x) = 1$ (-1) for $x \geq 0$ $(x < 0)$).

Given these data one can construct the corresponding QFT model. The fermions field algebra is defined as C^*–algebra generated by the field operators $\psi^*(f)$ linear in $f \in \mathcal{H}$ and $\psi(f) = \psi^*(f)^*$, obeying the CAR $(\psi^*(f) + \psi(g))^2 = (f, g)$ (inner product in $\mathcal{H}$). The physical representation of this algebra is then on the fermion Fock space $\mathcal{F}$ over $\mathcal{H}$ and is uniquely determined by the 'vacuum' Ω such that $\psi(f_+)\Omega = \psi^*(f_-)\Omega = 0$ for all $f_\pm \in \mathcal{H}_\pm$, which corresponds to the Dirac sea.

The aim then is to 'second quantize' observables and construct for operators u on $\mathcal{H}$ the corresponding multiparticle observables i.e. operators $\mathrm{d}\Gamma(u)$ on $\mathcal{F}$ such that $[\mathrm{d}\Gamma(u), \psi^*(f)] = \psi^*(uf)$. One then finds that it is not always possible to construct such an *operator* $\mathrm{d}\Gamma(u)$ but only if u is in g_1 introduced above. Thus the Schatten ideal conditions of NCG naturally appear here. The $\mathrm{d}\Gamma(u)$ for $u \in g_1$ form an algebra of operators on $\mathcal{F}$, and one has relations

$$[\mathrm{d}\Gamma(u), \mathrm{d}\Gamma(v)] = \mathrm{d}\Gamma([u, v]) + \hat{c}_1(u, v) \tag{8}$$

where

$$\hat{c}_1(u, v) = \frac{1}{2}\mathrm{Tr}_C\left(u[\varepsilon, v]\right) \tag{9}$$

is a term arising from the regularization (normal ordering) required in this construction. It is called Schwinger term in the physics literature and is a non–trivial 2–cocycle of the Lie algebra g_1.

As discussed below, this general framework is sufficient only for QFT in 1+1 dimensions since there the observables of interest actually are in g_1. In higher dimensions the interesting observables u are only in $g_{p\geq 2}$. Then $\mathrm{d}\Gamma(u)$ cannot be defined as operator. It is, however, still possible to define it as *sesquilinear form.* (Recall that for a s.l.f. A only transition amplitudes (f, Ag) are defined for f, g in some dense set of the Hilbert space.) To obtain suitable generalization of (8) one has to also consider splittings $\mathcal{H}$ given by other grading operators F, namely those for which $F - \varepsilon$ is in g_p. We denote this set as Gr_p. This is because the unitary operators $U = \exp(\mathrm{i}u) \in g_p$ generated by self–adjoint operators $u \in g_p$, act as transformations changing ε to $F = U^{-1}\varepsilon U$ which are in Gr_p, $F - \varepsilon = U^{-1}[\varepsilon, U]$. Infinitesimally this action is described by the Lie derivative $\hat{\mathcal{L}}_u f(F) \equiv -\mathrm{i}df(\exp(-\mathrm{i}tu)F\exp(\mathrm{i}tu))/\,dt|_{t=0}$ on quantities f depending on F. The construction of $\mathrm{d}\Gamma(u; F)$ can then be done by an additional 'wave function renormalization'. One obtains an algebra of operators $G(u, F) = \hat{\mathcal{L}}_u + \mathrm{d}\Gamma(u; F)$ similar to (8) with a Schwinger term $\hat{c}_{2p-1}(u, v; F)$ depending on $F \in Gr_p$, e.g. for $p = 2$ [5]

$$\hat{c}_3(u, v; F) = -\frac{1}{8}\mathrm{Tr}_C\left((F - \varepsilon)[[\varepsilon, u], [\varepsilon, v]]\right). \tag{10}$$

The case $p = 1$ is special since one can choose $\mathrm{d}\Gamma(u;F)$ and $\hat{c}_1$ independent of F and therefore can forget about Gr_1.

Gauss law anomalies. We now describe how the abstract framework of quasi–free second quantization above is used to derive the anomalous commutators of Gauss law generators for chiral QCD, i.e. chiral fermions coupled to a Yang–Mills field. Our setting is YM theory on $\mathbb{R}^d$ with structure group $\mathrm{SU}(N)$ represented by $N\times N$–matrices (for simplicity we do not distinguish $\mathrm{SU}(N)$ from its representation i.e. we assume $\mathrm{SU}(N) \subset \mathrm{gl}_N$). The space dimension is $d = 1, 3, 5 \ldots$. We denote as $\mathcal{A}_d$ the set of all YM field configuration, i.e. 1-forms $A = \sum_{i=1}^d A_i \mathrm{d}x^i$ with $A_i \in C_0^\infty(R^d, \mathrm{su}(N))$ ($\mathrm{su}(N) \subset \mathrm{gl}_N$ the Lie algebra of $\mathrm{SU}(N)$). The gauge group is $C_0^\infty(\mathbb{R}^d, \mathrm{SU}(N))$ and acts on $\mathcal{A}$ as $A \mapsto U \circ A \equiv U^{-1}AU - \mathrm{i}U^{-1}\mathrm{d}U$. Its Lie algebra is $C_0^\infty(\mathbb{R}^d, \mathrm{su}(N))$ acting on functionals f of A by the Lie derivative, $\mathcal{L}_X f(A) \equiv -\mathrm{i}df(\,\mathrm{e}^{\mathrm{i}tX} \circ A)/dt|_{t=0}$.

We now consider chiral fermions coupled to external Yang–Mills fields. Cohomological arguments show that for odd dimensions d there are 2–cocycles $c_d(X, Y; A)$, e.g.

$$c_1(X,Y) = \frac{1}{2\pi}\int_{\mathbb{R}^1} \mathrm{tr}_N\,(X\mathrm{d}Y) \tag{11}$$

for $d = 1$ [4] and

$$c_3(X,Y;A) = \frac{\mathrm{i}}{24\pi^2}\int_{\mathbb{R}^3} \mathrm{tr}_N\,(A[\mathrm{d}X,\mathrm{d}Y]) \tag{12}$$

for $d = 3$ [7]. It has been suggested on cohomological grounds that these 2–cocycles should arise as Schwinger terms commutators of the Gauss' law generators of chiral QCD [7]. This can be shown using the general formalism of QFSQ described above. (For a different solution to this problem for $d = 3$ see [8].)

The starting point is the 1-particle description of chiral fermions. The states at fixed time are described by the Hilbert space $h = L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu_{spin} \otimes \mathbb{C}^N_{color}$ where $\nu = 2^{(d-1)/2}$ is the number of spin indices. For a given YM configuration A, the 1-particle Hamiltonian is $\not{D}_A = \sum_{i=1}^d \gamma^i\,(-\mathrm{i}\partial_i + A_i)$ with γ^i the usual γ–matrices acting on $\mathbb{C}^\nu_{spin}$ and obeying $\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta^{ij}$. This naturally defines a self–adjoint operator on h for all $A \in \mathcal{A}_d$, and so do all $X \in C_0^\infty(\mathbb{R}^d; \mathrm{gl}_N)$ (containing $C_0^\infty(\mathbb{R}^d, \mathrm{SU}(N))$ and $C_0^\infty(\mathbb{R}^d, \mathrm{su}(N))$; we recall that every $X \in C_0^\infty(\mathbb{R}^d, \mathrm{gl}_N)$ defines a bounded operator on h, $(Xf)(x) = X(x)f(x)$ for all $f \in h$, which we denote by the same symbol).

The essential property now is that for h and $\varepsilon = \mathrm{sign}(\not{D}_0)$ above, there are natural embeddings of $C_0^\infty(\mathbb{R}^d, \mathrm{gl}_N)$ in g_p and of $\mathcal{A}_d$ in Gr_p,

$$X \in C_0^\infty(\mathbb{R}^d, \mathrm{gl}_N),\ A \in \mathcal{A}_d \Rightarrow X \in g_p,\ F_A \equiv \mathrm{sign}(\not{D}_A) \in Gr_p \text{ for } 2p = d+1. \tag{13}$$

Thus QFSQ gives by restriction the algebra of Gauss law generators of chiral QCD with Schwinger terms $\hat{c}_d(X, Y; F_A)$, and the question of whether the Schwinger terms (11) and (12) arise in chiral QCD reduces to the question whether the 2–cocycles (11) and (9) (for $d = 1$) and (12) and (10) (for $d = 3$) are cohomologous. This is very nontrivial since the abstract Schwinger terms $\hat{c}_d$ are given by highly non–local expressions whereas the Schwinger terms c_d are local integrals of de Rham forms. For $d = 1$ it has been known to be true since quite some time [2], and for $d = 3$ it was shown by explicit calculation recently [9]. However, given our discussion of graded differential algebras above, this result becomes very plausible since $\hat{c}_d$ is (up to a constant) just the noncommutative generalization of c_d. For $d = 1$ this is obvious, and for $d = 3$ it follows if we regard $F - \varepsilon$ as the noncommutative

generalization of A. The latter, however, is very natural: If the YM field is a pure gauge, $A = -\mathrm{i}U^{-1}\mathrm{d}U$ we have $F_A - \varepsilon = U^{-1}\varepsilon U - \varepsilon = -\mathrm{i}U^{-1}\hat{\mathrm{d}}U$. Equivalence of Schwinger terms thus is just a special case of a general embedding theorem of the de Rham complex $(\hat{\mathcal{C}}_p, \hat{\mathrm{d}}, \hat{\int})$ in the complex $(\mathcal{C}_d, \mathrm{d}, \int)$, especially that the noncommutative integral $\hat{\int}$ generalizes integration of de Rham forms $\int$,

$$(\mathrm{i})^d \mathrm{Tr}_C \left(\Gamma X_0[\varepsilon, X_1]\cdots[\varepsilon, X_d]\right) = c_d \int_{\mathbb{R}} \mathrm{tr}\left(X_0 \mathrm{d}X_1 \cdots \mathrm{d}X_d\right) \tag{14}$$

$\forall X_i \in C_0^\infty(\mathbb{R}^d; \mathrm{gl}_N)$, with some constants c_d and $\Gamma = 1$ for d odd and $\Gamma = \gamma_{d+1}$ for d even. A simple proof of this (motivated by the calculation in [9]) was recently given in [10] (one gets $c_d = (2\mathrm{i})^{[d/2]}2\pi^{d/2}/d(2\pi)^d\Gamma(d/2)$). For $d = 1$ this proves that the Schwinger terms are in fact identical (as $c_1 = 1/\pi$), for $d = 3$ it proves identity for pure YM field configurations which are pure gauges (as $c_3 = \mathrm{i}/3\pi^2$).

(3) Efficient Anomaly Calculations

In [9] we found that the calculus of pseudodifferential operators is an extremely powerful calculational tool in anomaly calculations: it gives a simple way of calculating regularized traces Tr_{reg} of commutators of operators (these methods were used earlier in a similar context in [8]). The same tool was essential in [10]. Recently we used this very tool for a very short QFT derivation of the axial anomalies for all even dimensions [11]. The strategy was as usual, to write the effective action for fermions in an external YM field as Tr_{reg} of a Dirac operator $\not{D}_A$ and calculate its variation under axial gauge transformations. Using only elementary rules for manipulating Hilbert space operators, we could write the latter as a sum of terms $\mathrm{Tr}_{reg}([a, b])$. To my opinion, this very short derivation makes mathematically precise traditional perturbative calculations of the anomaly and is very much in the spirit of NCG.

(4) Final Remarks

In my discussion in **(2)** I tried to convince the reader that the NCG viewpoint (i.e. generalizing from the differential algebra $(\mathcal{C}_d, \mathrm{d}, \int)$ to $(\hat{\mathcal{C}}_p, \hat{\mathrm{d}}, \hat{\int})$) is very useful for general QFT calculations concentrating on the essential QFT aspects, namely the nature of the UV divergences characteristic for a specific dimension. It naturally leads to generalizations of Schwinger terms to the noncommutative setting. This and the anomaly calculation discussed in **(3)** suggest that all YM fermion anomalies should have noncommutative generalizations. I recently found that this is indeed the case. In fact, there is a noncommutative generalization of the whole tower of descent equations [12]. It involves a generalization of the noncommutative integration formula (14) to $(d - n)$–dimensional submanifolds of $\mathbb{R}^d$ such that Stokes' theorem holds (details will appear elsewhere). To my opinion, this noncommutative version of the descent equations provides a nice explanation of how the rich geometric structure of anomalies arises from QFT: it is present already on the level of Hilbert space operators entering the Feynman diagrams.

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